NUMEROUS SERIES (II - PART)

D`Alember criteria

If for series $\sum_{n=1}^{\infty} a_n$ there is $\overline{\lim_{n\to\infty}}\left|\frac{a_{n+1}}{a_n}\right| = r$ then :

- For r > 1 series is divergent
- For r = 1 undecidable
- For r < 1 series is convergent

Example 1.

Examine the convergence of series $\sum_{n=1}^{\infty} \frac{1}{n!}$

Solution:

First, to determine a_n. Here it is $a_n = \frac{1}{n!}$ (Ie take all the behind the label of the series). Further determined a_{n+1}. How?

Watch a_n and instead n we put *n+1* , so $a_{n+1} = \frac{1}{(n+1)!}$

Now we use D`Alember criteria:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}} = \lim_{n\to\infty}\frac{1}{(n+1)} = 0$$

So, we got r = 0 < 1, and by this criteria, series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Example 2.

Examine the convergence of series $\sum_{n=1}^{\infty} \frac{2^n}{n}$

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}}\right| = \lim_{n\to\infty}\frac{n\cdot 2^{n+1}}{2^n\cdot(n+1)} = \lim_{n\to\infty}\frac{n\cdot \cancel{2^n}\cdot 2}{\cancel{2^n}\cdot(n+1)} = 2\lim_{n\to\infty}\frac{n}{n+1} = 2\cdot 1 = 2$$ Here we get that r = 2, which tells us that $\sum_{n=1}^{\infty} \frac{2^n}{n}$ diverges

Example 3.

Examine the convergence of series $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2^{n+1}}$

Solution:

Here is $a_n = \frac{(2n-1)!!}{(2n)!!}\frac{1}{2^{n+1}}$, And to remind ourselves what this two-factorial means…

$$n! = n(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1$$
$$n!! = n(n-2)\cdot(n-4)\cdot \ldots \cdot$$

Depending on whether n is odd or even, when there is !! we get to 2 or 1. For example:

$$10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$$

$$9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$$

Here, we encourage you to keep track of the brackets , because $(n!)! \neq n!!$

To get back to the task:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(2n+1)!!}{(2n+2)!!}\cdot\frac{1}{2^{n+2}}}{\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2^{n+1}}} = \lim_{n\to\infty} \frac{2^{n+1}}{2^{n+2}}\cdot\frac{(2n+1)!!}{(2n-1)!!}\cdot\frac{(2n)!!}{(2n+2)!!} =$$

$$= \lim_{n\to\infty}\frac{\cancel{2^{n+1}}}{\cancel{2^{n+1}}\cdot 2}\cdot\frac{(2n+1)\,\cancel{(2n-1)!!}}{\cancel{(2n-1)!!}}\cdot\frac{\cancel{(2n)!!}}{(2n+2)\,\cancel{(2n)!!}}$$

$$= \lim_{n\to\infty}\frac{1}{2}\cdot\frac{2n+1}{2n+2} = \boxed{\frac{1}{2}}$$

So , r = ½, then given series converge.

Example 4.

Examine the convergence of series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{\cancel{(n+1)}\cdot\cancel{n}!}{\cancel{n}!}\cdot\frac{n^n}{\cancel{(n+1)}\,(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n =$$

$$= \lim_{n\to\infty}\left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}$$

Let's solve this in another way.

We will use something which is often used when there is $n!$. This is the so-called Stirling approximation:

$$\boxed{n! \approx \sqrt{2n\pi} \cdot n^n \cdot e^{-n}}$$

Now we have:

$$a_n = \frac{n!}{n^n} \sim \frac{\sqrt{2n\pi} \cdot \cancel{n^n} \cdot e^{-n}}{\cancel{n^n}} = \frac{\sqrt{2n\pi}}{e^n}$$

Try again this criteria:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{\sqrt{2(n+1)\pi}}{e^{n+1}}}{\frac{\sqrt{2n\pi}}{e^n}} = \lim_{n\to\infty} \frac{e^n}{e^{n+1}} \cdot \frac{\sqrt{2(n+1)\pi}}{\sqrt{2n\pi}} = \lim_{n\to\infty} \frac{\cancel{e^n}}{\cancel{e^n} \cdot e} \cdot \sqrt{\frac{\cancel{2\pi}(n+1)}{\cancel{2\pi} n}} =$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \underbrace{\boxed{\sqrt{\frac{(n+1)}{n}}}}_{teži\ 1} = \boxed{\frac{1}{e}}$$

So, r = 1/e , this series converges.

Example 5.

Examine the convergence of series $\sum_{n=1}^{\infty} \frac{n^p}{n!}$

Solution:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^{p+1}}{(n+1)!}}{\frac{n^p}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{p+1}}{n^p} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)^p \cancel{(n+1)}}{n^p} \frac{\cancel{n!}}{\cancel{(n+1)}\cancel{n!}} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^p =$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^p = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^{n \frac{p}{n}} = e^{\lim\limits_{n\to\infty} \frac{p}{n}} = e^o = 1$$

Criterion is undecidable, apply an approximation:

$$a_n = \frac{n^p}{n!} \sim \frac{n^p}{\sqrt{2n\pi} \cdot n^n \cdot e^{-n}} = \frac{n^p \cdot e^n}{\sqrt{2n\pi} \cdot n^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{(n+1)^p\cdot e^{n+1}}{\sqrt{2(n+1)\pi}\cdot(n+1)^{n+1}}}{\dfrac{n^p\cdot e^n}{\sqrt{2n\pi}\cdot n^n}} = \lim_{n\to\infty}\frac{\sqrt{2n\pi}}{\sqrt{2(n+1)\pi}}\frac{e^{n+1}}{e^n}\frac{n^n}{(n+1)^{n+1}}\frac{(n+1)^p}{n^p} =$$

$$\lim_{n\to\infty}\underbrace{\boxed{\frac{\sqrt{2n\pi}}{\sqrt{2(n+1)\pi}}}}_{teži\ 1}\frac{\cancel{e^n}\cdot e}{\cancel{e^n}}\frac{n^n}{(n+1)^n\cdot(n+1)}\frac{(n+1)^p}{n^p} =$$

$$\lim_{n\to\infty} e\cdot\left(\frac{n}{n+1}\right)^n\frac{1}{n+1}\underbrace{\boxed{\left(\frac{n+1}{n}\right)^p}}_{teži\ 1} = \lim_{n\to\infty} e\cdot\left(1+\frac{n}{n+1}-1\right)^n\frac{1}{n+1} = \lim_{n\to\infty} e\cdot\left(1+\frac{n-n-1}{n+1}\right)^n\frac{1}{n+1}$$

$$= \lim_{n\to\infty} e\cdot\left(1+\frac{-1}{n+1}\right)^n\frac{1}{n+1} = e\lim_{n\to\infty}\left(1+\frac{1}{-n-1}\right)^{n\cdot(-n-1)\cdot\frac{1}{(-n-1)}}\frac{1}{n+1} =$$

$$= e\boxed{\lim_{n\to\infty}\left(1+\frac{1}{-n-1}\right)^{(-n-1)\cdot\frac{n}{(-n-1)}}}\frac{1}{n+1} = e\cdot e^{-1}\lim_{n\to\infty}\frac{1}{n+1} = 0$$

So, given series converge.

<u>Roots Cauchy criteria</u> :

If for series $\sum_{n=1}^{\infty} a_n$ there is $\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} = p$ then :

- For $p > 1$ series is divergent
- For $p = 1$ undecidable
- For $p < 1$ series is convergent

<u>Example 6.</u>

Examine the convergence of series $\sum_{n=1}^{\infty}\left(\frac{n-1}{n+1}\right)^{n(n-1)}$

<u>Solution:</u>

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}}=\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{\frac{n(n-1)}{n}}=\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{n-1}=$$

$$\lim_{n\to\infty}\left(1+\frac{n-1}{n+1}-1\right)^{n-1}=\lim_{n\to\infty}\left(1+\frac{n-1-n-1}{n+1}\right)^{n-1}=\lim_{n\to\infty}\left(1+\frac{-2}{n+1}\right)^{n-1}=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n+1}{-2}}\right)^{\frac{n+1}{-2}\cdot\frac{-2}{n+1}(n-1)}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n+1}{-2}}\right)^{\frac{n+1}{-2}\cdot\frac{-2}{n+1}(n-1)}=e^{\lim_{n\to\infty}\frac{-2n+2}{n+1}}=e^{-2}=\boxed{\frac{1}{e^2}}$$

As is $r=\frac{1}{e^2}<1$, series $\sum_{n=1}^{\infty}\left(\frac{n-1}{n+1}\right)^{n(n-1)}$ converge by Cauchy criteria.

Example 7.

Examine the convergence of series $\sum_{n=1}^{\infty}\left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}$

Solution :

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{1+\cos n}{2+\cos n}\right)^{n(2-\frac{\ln n}{n})}}=\lim_{n\to\infty}\left(\frac{1+\cos n}{2+\cos n}\right)^{2-\frac{\ln n}{n}}$$

We know that $\frac{\ln n}{n}$ approaches 0 when n tends infinity , and *cosn* can not have more value from 1.Then:

$$\lim_{n\to\infty}\left(\frac{1+\cos n}{2+\cos n}\right)^{2-\frac{\ln n}{n}}\le\lim_{n\to\infty}\left(\frac{1+1}{2+1}\right)^{2-0}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^{2}=\frac{4}{9}<1$$

So, this series converges.

Raabe criteria :

If for series $\sum_{n=1}^{\infty}a_n$ there is $\lim_{n\to\infty}n(\frac{a_n}{a_{n+1}}-1)$ = t then :

- For $t>1$ series is convergent
- For $t=1$ undecidable
- For $t<1$ series is divergent

Example 8.

Examine the convergence of series $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$

Solution :

$$\lim_{n\to\infty} n(\frac{a_n}{a_{n+1}}-1) = \lim_{n\to\infty} n(\frac{\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}}{\frac{(2n+1)!!}{(2n+2)!!}\frac{1}{2n+3}}-1) = \lim_{n\to\infty} n(\frac{(2n-1)!!}{(2n+1)!!}\frac{(2n+2)!!}{(2n)!!}\frac{2n+3}{2n+1}-1)$$

$$= \lim_{n\to\infty} n(\frac{(2n-1)!!}{(2n+1)(2n-1)!!}\frac{(2n+2)(2n)!!}{(2n)!!}\frac{2n+3}{2n+1}-1) =$$

$$= \lim_{n\to\infty} n(\frac{(2n+2)(2n+3)}{(2n+1)^2}-1) = \lim_{n\to\infty} n(\frac{(2n+2)(2n+3)-(2n+1)^2}{(2n+1)^2}) =$$

$$= \lim_{n\to\infty} n(\frac{4n^2+6n+4n+6-4n^2-4n-1}{(2n+1)^2}) = \lim_{n\to\infty} n(\frac{6n+5}{(2n+1)^2}) =$$

$$= \lim_{n\to\infty} \frac{6n^2+5n}{4n^2+4n+1} = \frac{6}{4} = \boxed{\frac{3}{2}>1}$$

This means that this series , by this criteria converge.

Example 9.

Find the value of the parameter p so that $\sum_{n=1}^{\infty} \frac{n!e^n}{n^{n+p}}$ converges.

Solution:

First we handle the term $\frac{a_n}{a_{n+1}}$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{n!e^n}{n^{n+p}}}{\frac{(n+1)!e^{n+1}}{(n+1)^{n+1+p}}} = \frac{n!}{(n+1)!}\frac{e^n}{e^{n+1}}\frac{(n+1)^{n+1+p}}{n^{n+p}} = \frac{n!}{(n+1)\cdot n!}\frac{e^n}{e^n\cdot e}\frac{(n+1)^{n+p}\cdot(n+1)}{n^{n+p}} = \frac{1}{e}\frac{(n+1)^{n+p}}{n^{n+p}} =$$

$$= \frac{1}{e}\left(\frac{n+1}{n}\right)^{n+p} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+p}$$

Now , we use :

$e^{\ln\Theta} = \Theta,$ where is $\Theta=\left(1+\frac{1}{n}\right)^{n+p}$

$$\left(1+\frac{1}{n}\right)^{n+p} = e^{\ln\left(1+\frac{1}{n}\right)^{n+p}} = e^{(n+p)\ln\left(1+\frac{1}{n}\right)}$$

We have:

$$\frac{a_n}{a_{n+1}} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+p} = \frac{1}{e}e^{(n+p)\ln\left(1+\frac{1}{n}\right)} = e^{-1}\cdot e^{(n+p)\ln\left(1+\frac{1}{n}\right)} = e^{-1+(n+p)\ln\left(1+\frac{1}{n}\right)}$$

For $\ln\left(1+\frac{1}{n}\right)$ we will use : $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$, $-1<x<1$

$\ln(1+\frac{1}{n}) = \frac{1}{n} - \frac{1}{2n^2} + o(\frac{1}{n^2})$ now:

$$\frac{a_n}{a_{n+1}} = e^{-1+(n+p)\ln\left(1+\frac{1}{n}\right)} = e^{-1+(n+p)(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))} = e^{-1+1-\frac{1}{2n}+\frac{p}{n}-\frac{p}{2n^2}+o(\frac{1}{n^2})} = e^{\frac{p}{n}-\frac{1}{2n}+o(\frac{1}{n})} = e^{\frac{p-\frac{1}{2}}{n}+o(\frac{1}{n})} = 1+\frac{p-\frac{1}{2}}{n}+o(\frac{1}{n}) \text{ when } n\to\infty$$

Raabe criteria:

$$\lim_{n\to\infty} n(\frac{a_n}{a_{n+1}}-1) = \lim_{n\to\infty} n(1+\frac{p-\frac{1}{2}}{n}-1) = \lim_{n\to\infty} n\frac{p-\frac{1}{2}}{n} = p-\frac{1}{2}$$

Now, we think:

$p-\frac{1}{2} > 1 \to p > \frac{3}{2}$ series converges.

Cauchy integral criteria

If the function f (x) decreases, it is continuous and positive , then series $\sum_{n=1}^{\infty} f(n)$ convergent or divergent simultaneously with integral $\int_1^{\infty} f(x)dx$

Example 10.

Examine the convergence of series $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$

Solution:

We are looking at the integral: $\int_1^{\infty} \frac{1}{x^{\alpha}} dx$

$$\int_1^{\infty} \frac{1}{x^{\alpha}} dx = \lim_{A\to\infty} \int_1^{A} \frac{1}{x^{\alpha}} dx = \lim_{A\to\infty} \int_1^{A} x^{-\alpha} dx = \lim_{A\to\infty} \frac{x^{-\alpha+1}}{-\alpha+1} \Big/_1^A = \lim_{A\to\infty} \left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right) =$$

i) If $\alpha>1$ then $\lim_{A\to\infty}\left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right) = 0 - \frac{1}{-\alpha+1} = \frac{1}{\alpha-1}$

ii) If $\alpha \le 1$ then $\lim_{A\to\infty}\left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right) = \infty$

Thus, the series converges for $\alpha>1$, and diverges for $\alpha \le 1$.

Example 11.

Examine the convergence of series if $a_n = \frac{1}{n \ln^p n}$ where is n>1

Solution:

$$\int_2^{\infty} \frac{1}{x \ln^p x} dx = \lim_{A\to\infty} \int_2^{A} \frac{1}{x \ln^p x} dx$$

First we solve the integral on the side without a borders:

$$\int \frac{1}{x \ln^p x} dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \end{array} \right| = \int \frac{1}{t^p} dt = \int t^{-p} dt = \frac{t^{-p+1}}{-p+1} = \frac{t^{1-p}}{1-p}$$

$$\int_2^{\infty} \frac{1}{x \ln^p x} dx = \lim_{A\to\infty} \int_2^{A} \frac{1}{x \ln^p x} dx = \lim_{A\to\infty} \frac{(\ln x)^{1-p}}{1-p} \Big/_2^A = \lim_{A\to\infty} \frac{(\ln A)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} =$$

i) If $1-p < 0 \to p>1$ converges $\frac{(\ln A)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} = 0 - \frac{(\ln 2)^{1-p}}{1-p} = \frac{(\ln 2)^{1-p}}{p-1}$

ii) If $p<1$ diverges $\lim_{A\to\infty} \frac{(\ln A)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} = \infty$

Gauss criteria

If for series $\sum_{n=1}^{\infty} a_n$ there is

$$\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + o(\frac{1}{n^{1+\varepsilon}}) \quad \text{for} \quad \forall \varepsilon > 0 \ \text{then :}$$

i) If $\lambda > 1$ series is convergent
ii) If $\lambda < 1$ series is divergent
iii) If $\lambda = 1$ then $\left\{ \begin{array}{l} \text{for } \mu > 1 \text{ is convergent} \\ \text{for } \mu < 1 \text{ is divergent} \end{array} \right\}$

Example 12.

Examine the convergence of series $\sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} \right]^p$

Solution:

$$\frac{a_n}{a_{n+1}} = \frac{\left[\frac{(2n-1)!!}{(2n)!!} \right]^p}{\left[\frac{(2n+1)!!}{(2n+2)!!} \right]^p} = \left[\frac{(2n-1)!!}{(2n+1)!!} \frac{(2n+2)!!}{(2n)!!} \right]^p = \left[\frac{(2n-1)!!}{(2n+1)(2n-1)!!} \frac{(2n+2)(2n)!!}{(2n)!!} \right]^p = \left[\frac{2n+2}{2n+1} \right]^p$$

Now pack a little the term and we use binomial formula:

$$\left[\frac{2n+2}{2n+1} \right]^p = \left[\frac{2n+1+1}{2n+1} \right]^p = \left[1 + \frac{1}{2n+1} \right]^p =$$

$$= \binom{p}{0} 1^p (\frac{1}{2n+1})^0 + \binom{p}{1} 1^{p-1} (\frac{1}{2n+1})^1 + \binom{p}{2} 1^{p-2} (\frac{1}{2n+1})^2 + \ldots$$

$$= 1 + \frac{p}{2n+1} + \underline{\left| \frac{p(p+1)}{2(2n+1)^2} + o(\frac{1}{n^2}) \right.}$$

$$= 1 + \frac{p}{2n+1} + o(\frac{1}{n^2})$$

$$= 1 + \frac{p}{2(n+\frac{1}{2})} + o(\frac{1}{n^2})$$

$$= 1 + \frac{p/2}{n+1/2} + o(\frac{1}{n^2}) \ \text{ when } \ n \to \infty$$

$$= 1 + \frac{p/2}{n} + o(\frac{1}{n^2})$$

This compares with $\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + o(\frac{1}{n^{1+\varepsilon}})$

It is clear that $\lambda = 1$ and we need $\mu = \frac{p}{2}$

i) If $\mu = \frac{p}{2} > 1 \rightarrow p > 2$ series converges

ii) If $\mu = \frac{p}{2} < 1 \rightarrow p < 2$ series diverges